Problem Set: The Ramsey–Cass–Koopmans Model

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

Instructions. Answer all questions. Show all mathematical derivations clearly. Answers without derivation receive limited credit. Time is discrete, $t = 0, 1, 2, \ldots$. There is no population growth and no depreciation. Technology evolves as

$$A_{t+1} = (1+g)A_t, \qquad g > 0.$$

There are L identical infinitely lived households. Define

$$k_t \equiv \frac{K_t}{A_t L}, \qquad \widetilde{c}_t \equiv \frac{c_t}{A_t}.$$

Firms operate a CRS production function

$$Y_t = F(K_t, A_t L_t^D) = A_t L_t^D f(k_t),$$

where $f'(k) > 0$, $f''(k) < 0$, and $\lim_{k \to 0} f'(k) = +\infty$.

Question 1: Competitive Equilibrium and the Planner Problem [Total: 60 marks]

The representative household has lifetime utility

$$U(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1,$$

where $u'(c) > 0$, $u''(c) < 0$, and $\lim_{c \to 0} u'(c) = +\infty$. The household budget constraint is

$$c_t + s_t = A_t w_t + (1 + r_t)s_{t-1}, \qquad s_{-1} = K_0/L.$$

(a) Derive the firm's first-order conditions

$$r_t = f'(k_t), \qquad w_t = f(k_t) - f'(k_t)k_t,$$

and show why CRS and perfect competition imply zero profits.

(b) Derive the household Euler equation

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1}.$$

Then explain why the Inada condition gives an interior consumption path.

(c) Use market clearing,

$$L_t^D = L, \qquad K_{t+1} = L s_t,$$

to derive the RCK transition equations:

$$\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1},$$
$$\frac{u'(A_t \widetilde{c}_t)}{\beta u'(A_{t+1}\widetilde{c}_{t+1})} = 1 + f'(k_{t+1}).$$

For CRRA utility,

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}, \qquad \theta > 0,$$

derive

$$\left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right)^{\theta} = \frac{\beta[1 + f'(k_{t+1})]}{(1+g)^{\theta}}.$$

(d) Write down the social planner's problem and derive its first-order condition. Show that the planner's equations coincide with the competitive-equilibrium transition equations. What does this imply about Pareto efficiency in the RCK model?

Question 2: Steady State, Golden Rule, and Saddle-Path Stability [Total: 40 marks]

Assume CRRA utility and Cobb–Douglas production:

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}, \qquad f(k) = k^{\alpha}, \qquad 0 < \alpha < 1.$$

(a) Derive the steady-state conditions

$$\widetilde{c}^* = f(k^*) - g k^*, \qquad \beta[1 + f'(k^*)] = (1+g)^{\theta}.$$

Solve for k^* under Cobb–Douglas production and interpret k^* as the modified Golden Rule capital stock.

(b) Derive the Golden Rule capital stock from

$$\widetilde{c} = f(k) - gk.$$

Show that

$$f'(k_{\text{GR}}) = g, \qquad k_{\text{GR}} = \left(\frac{\alpha}{g}\right)^{1/(1-\alpha)}.$$

Under the well-defined-utility restriction

$$\beta(1+g)^{1-\theta} < 1,$$

prove that $k^* < k_{\mathrm{GR}}$.

(c) For

$$\alpha = \frac{1}{3}, \qquad \beta = 0.96, \qquad \theta = 2, \qquad g = 0.02,$$

compute k^*, y^*, $\widetilde{c}^*$, r^*, k_{GR}, and $\widetilde{c}_{\mathrm{GR}}$. Compare k^* with k_{GR}.

(d) Rewrite the RCK system as a first-order dynamic system in $(k_t, \widetilde{c}_t)$. Derive the $\Delta k = 0$ and $\Delta\widetilde{c} = 0$ loci. Then linearise the system around $(k^*, \widetilde{c}^*)$ and show that the steady state is saddle-path stable. Explain the role of the transversality condition.